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Borel-type bounds for the self-avoiding walk connective constant

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Abstract

Let μ be the self-avoiding walk connective constant on $\mathbb{Z}^d$. We show that the asymptotic expansion for $\beta_c = 1/\mu$ in powers of $1/(2d)$ satisfies Borel-type bounds. This supports the conjecture that the expansion is Borel summable.

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1. Introduction

Let $\mathbb{Z}^d$ denote the hypercubic lattice, with nearest neighbour edges. A self-avoiding walk of length n is a sequence of points $\omega_0, \omega_1, \ldots, \omega_n$ in $\mathbb{Z}^d$ such that $|\omega_i - \omega_{i+1}| = 1$ and for $i \neq j$, $\omega_i \neq \omega_j$. Let c_n denote the number of self-avoiding walks, up to translation invariance, of length n on $\mathbb{Z}^d$. It is well known that the limit $\mu(d) = \lim_{n\to\infty} c_n^{1/n}$ exists [1]; the limit is called the connective constant. Fisher and Gaunt calculated that [2]

$$\mu = 2d - 1 - 1/(2d) - 3/(2d)^2 - 16/(2d)^3 - 102/(2d)^4 - \cdots.$$

However, their calculation is somewhat mysterious. Firstly, they leave open the question of whether or not the expansion can be continued to higher orders of $1/d$. Secondly, even though the error term '. . .' is left uncontrolled, numerical extrapolation techniques yield surprisingly accurate estimates for μ.

Expansions in powers of $1/d$ have been developed for many other models in statistical physics, such as the Ising model [2], percolation [3], lattice animals [4] and the n-vector model [5]. Finding the coefficients of the expansion is normally computationally intensive. It is often even more difficult to determine the basic properties of the expansion. What is the radius of convergence? Is it an asymptotic expansion? Can the expansion be interpreted as a Borel sum?

The self-avoiding walk is most easily understood in high dimensions. As $d \to \infty$, paths in $\mathbb{Z}^d$ with large loops become relatively rare. It is therefore useful to consider a walk with only local self-avoidance. Say that $\omega_0, \ldots, \omega_n$ is a memory-τ self-avoiding walk if $\omega(i) \neq \omega(j)$

for $0 < |i - j| \leqslant \tau$. Let $c_n^{(\tau)}$ denote the number of n-step memory-τ walks, up to translation invariance, and let $\mu_\tau(d) = \lim_{n\to\infty}(c_n^{(\tau)})^{1/n}$. Using memory-4 self-avoiding walks as a starting point (taking into account loops of size 2 and 4) Kesten showed that [6]

$$\mu(d) = 2d - 1 - 1/(2d) + \mathrm{O}(1/d^2).$$

By considering finite-memory self-avoiding walks with longer memories, the order of the error bound can be improved. However, not only is this method extremely computationally taxing, it also provides no guarantee that the resulting expansion will only contain integer powers of d.

The series expansion for μ was placed on a much firmer footing by Hara and Slade using the lace expansion [7]. The lace expansion is a powerful technique for exploring the properties of the self-avoiding walk in dimensions $d > 4$; we refer the reader to [8] for a recent introduction. Hara and Slade showed that the connective constant μ has an asymptotic expansion in integer powers of $1/(2d)$ to all orders, with all the coefficients taking integer values.

We will actually phrase their result in terms of the series expansion for the reciprocal of $\mu(d)$. The quantity $\beta_c = 1/\mu(d)$ is the radius of convergence of the self-avoiding walk susceptibility $\chi(z) = \sum c_n z^n$. For convenience, set $s = 1/(2d)$. Hara and Slade showed that there are constants (α_n) such that for $M = 1, 2, \ldots,$ [7],

$$\beta_c(s) = \sum_{n=1}^{M-1} \alpha_n s^n + \mathrm{O}(s^M). \tag{1.1}$$

They also verified rigorously that the first six terms in the expansion match the exact calculation of Fisher and Gaunt.

The lace expansion can be used to automate the process of calculating the coefficients of the asymptotic expansion for β_c and μ. The computational complexity of the process is reduced using a combinatorial trick known as the two-step method. Using a supercomputer to implement the two-step method, the first 13 coefficients of β_c have been found [9],

$$1, 1, 2, 6, 27, 157, 1065, 7865, 59\,665, 422\,421, 1991\,163, -16\,122\,550, -805\,887\,918. \tag{1.2}$$

It is not known, but it is widely believed, that the radius of convergence of the expansion for β_c is zero. We will show that the partial sums satisfy Borel-type bounds. Borel summability raises the prospect of calculating μ from the series expansion even if the radius of convergence is zero.

Theorem 1.3. *There exists a constant C_1 such that for all d*

$$\left|\beta_c(s) - \sum_{n=1}^{M-1} \alpha_n s^n\right| \leqslant C_1^M s^M M!, \qquad M = 1, 2, \ldots. \tag{1.3}$$

The motivation for theorem 1.3 is discussed in section 2. In section 4, we use the lace expansion to derive a formula for the α_n. This formula is used in section 5 to control the growth of the α_n as $n \to \infty$. In section 6, we consider the diagrammatic estimates for the lace expansion. Finally, in section 7 we prove theorem 1.3.

2. Borel summability and the spherical model

In light of theorem 1.3, it is natural to ask if β_c can be recovered from α_n by means of a Borel sum. Let B denote the Borel transform of the asymptotic expansion for β_c; B is well defined (see lemma 5.1) in a neighbourhood of zero by

$$B(t) = \sum_{n=1}^{\infty} \alpha_n t^n / n!.$$

We conjecture that B can be extended analytically to a neighbourhood of the positive real axis, and that $\beta_c(s)$ is equal to the Borel sum

$$\sum_{\text{Borel}} \alpha_n s^n := \frac{1}{s} \int_0^{\infty} e^{-t/s} B(t)\, dt.$$

There are two reasons for making this conjecture. Firstly, with $R > 0$, let $C_R := \{z \in \mathbb{C} : \operatorname{Re} z^{-1} > R^{-1}\}$ denote the open disc in $\mathbb{C}$ with the centre $R/2$ and the diameter R. Suppose that β_c can be extended to an analytic function on C_R such that (1.3) holds for all $s \in C_R$. Under this assumption, the Borel sum is well defined in C_R and equal to β_c [10]. Unfortunately, it is not clear how to extend β_c to an analytic function on C_R. Interpreting the Borel sum remains an open problem.

Secondly, there is the case of the spherical model, which is a spin system defined on $\mathbb{Z}^d$. There is a surprising connection between the spherical model and self-avoiding walk; both are identified with limits of the n-vector models (also known as the O(n) model). The n-vector model is defined for positive integer values of n; for example, the Ising model corresponds to $n = 1$. The model has been studied extensively by scientists; many aspects of the models have been solved 'exactly' [11]. De Gennes ([12] and [1, section 2.3]) showed that in an abstract sense, the self-avoiding walk can be viewed as the 'limit' of the n-vector model as $n \to 0$. Stanley showed that as $n \to \infty$, the free energy of the n-vector model approaches the spherical model free energy [13]. The spherical model is thus said to be the limit as $n \to \infty$ of the n-vector model.

There is an exact solution $K_c(d)$ for the critical point of the spherical model. Gerber and Fisher show that $K_c(d)$ can be written as a $1/d$ expansion [5]; it is a rare example of a $1/d$ expansion about which a great deal is known. They prove that while the radius of convergence of the expansion is zero, the expansion can be interpreted as a Borel sum [5, (2.14)]:

$$K_c(d) = \sum_{\text{Borel}} \frac{K_n}{(2d)^n} \quad \text{with Borel transform} \quad \sum_{n=1}^{\infty} \frac{K_n x^n}{n!}.$$

Note that the signs of the coefficients (K_n) oscillate. The first 12 coefficients are positive, the next 8 are negative, the next 9 are positive; the pattern of signs goes

12, 8, 9, 9, 9, 9, 9, 9, 9, 10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, 9, 9, 9, 9, 9, ...

This oscillation is related to the fact that the Borel transform has no poles on the positive real axis. We saw in (1.2) that the coefficients α_n for the self-avoiding walk also show a change of sign. The first 11 are positive; α_{12} and α_{13} are negative.

3. Notation

Given a generating function $\phi(\beta)$, we will write $[\beta^n]\phi(\beta)$ to denote the coefficient of β^n.

We will refer to the fact that $(n/e)^n \leqslant n! \leqslant n^n$ for $n = 0, 1, 2 \ldots$ as Stirling's approximation.

4. From lace expansions to asymptotic expansions

The lace expansion can be thought of as a sum of inclusion/exclusion terms. For a derivation of the lace expansion, see [8, section 3.2]. The finite memory self-avoiding walks will play a vital role in the proof of theorem 1.3. For a derivation of the lace expansion for memory-τ self-avoiding walk, see [14].

A *lace* of type N and length a is a sequence of open intervals $(s_1, t_1), \ldots, (s_N, t_N)$ such that

(i) s_i and t_i are integers with $0 < t_i - s_i \leqslant \tau$,
(ii) $s_1 = 0$ and $t_N = a$,
(iii) for $i = 1, \ldots, N-1$, (s_i, t_i) intersects (s_{i+1}, t_{i+1}), and
(iv) if $|i - j| > 1$, (s_i, t_i) and (s_j, t_j) are disjoint.

For example, $(0, 4), (3, 5), (4, 6)$ is a lace if $\tau \geqslant 4$.

A simple walk $\omega(0), \omega(1), \ldots, \omega(a)$ starting from 0 is said to be *compatible* with the lace $(s_1, t_1), \ldots, (s_N, t_N)$ if each interval corresponds to a loop:

$$\omega(s_i) = \omega(t_i) \qquad \text{for} \qquad i = 1, \ldots, N,$$

and if certain self-avoidance constraints are satisfied.

(i) $\omega(0), \ldots, \omega(t_1 - 1)$ is a memory-τ self-avoiding walk.
(ii) $\omega(1), \ldots, \omega(t_2 - 1)$ is a memory-τ self-avoiding walk (for $N \geqslant 2$).
(iii) $\omega(t_{i-2}), \ldots, \omega(t_i - 1)$ is a memory-τ self avoiding walk (for $3 \leqslant i \leqslant N$).

Let $\pi_a^{(N)}(x; \tau)$ count the number of a-step simple walks from 0 to x that are compatible with a memory-τ type-N lace. The lace expansion is defined as

$$\Pi_\beta(x; \tau) = \sum_{N=1}^{\infty} (-1)^N \Pi_\beta^{(N)}(x; \tau) \qquad \text{where} \qquad \Pi_\beta^{(N)}(x; \tau) = \sum_{a=N+1}^{\infty} \pi_a^{(N)}(x; \tau) \beta^a.$$

The Fourier transform for functions $f : \mathbb{Z}^d \to \mathbb{R}$ is given by

$$\hat{f}(k) = \sum_x f(x)\, \mathrm{e}^{-\mathrm{i}k\cdot x}, \qquad k \in [-\pi, \pi]^d.$$

We will write $\hat{\pi}_a^{(N)}(k; \tau)$, $\hat{\Pi}_\beta^{(N)}(k; \tau)$ and $\hat{\Pi}_\beta(k; \tau)$ for the Fourier transforms of $\pi_a^{(N)}(x; \tau)$, $\Pi_\beta^{(N)}(x; \tau)$ and $\Pi_\beta(x; \tau)$, respectively.

The starting point in our analysis will be [7, (2.2)]. Let $\beta_\tau = 1/\mu_\tau$, and take $\beta_\infty = \beta_\mathrm{c}$. When d is sufficiently large, for τ finite and $\tau = \infty$,

$$\beta_\tau = s(1 - \hat{\Pi}_{\beta_\tau}(0; \tau)). \tag{4.1}$$

In this section, we will use this formula to derive series expansions for β_τ.

The definition of the lace expansion respects the symmetries of the underlying lattice. There are $2^d d!$ ways of choosing an ordered orthonormal basis for $\mathbb{R}^d$ from the set $\mathbb{Z}^d$. Each simple walk in $\mathbb{Z}^d$ with dimensionality D is equivalent to $2d(2d-2)\cdots(2d-2D+2)$ other walks under the action of this group of symmetries.

Let $f_\tau(a, N, D)$ count the number of equivalence classes of the set of simple walks in $\mathbb{Z}^D$ that have dimensionality D, length a, and are compatible with memory-τ laces of type N. If $a < 2D$, $f_\tau(a, N, D) = 0$. Therefore, we can write the number of walks compatible with memory-τ laces of length a and type N in $\mathbb{Z}^d$ as a polynomial in powers of $s^{-1} = 2d$:

$$\sum_{D=1}^{\lfloor a/2 \rfloor} f_\tau(a, N, D) 2d(2d-2)\cdots(2d-2D+2) = \sum_{b=\lceil a/2 \rceil}^{a-1} c_{a,b,N} s^{b-a}.$$

Let $I = \{(a,b) : b = 1, 2, \ldots; a = b+1, \ldots, 2b\}$ and set

$$c_{a,b} = \sum_{N=1}^{\infty} (-1)^{N+1} c_{a,b,N}, \qquad (a,b) \in I.$$

The $c_{a,b}$ depends implicitly on τ, but $c_{a,b}$ is fixed once $\tau \geqslant a$. Using this notation to rewrite (4.1) yields a formal power series,

$$\begin{aligned} \beta_\tau &= s \left[1 + \sum_{(a,b)\in I} \beta_\tau^a c_{a,b} s^{b-a} \right] \\ &= s\left[1 + \beta_\tau^2 c_{2,1} s^{-1} + \beta_\tau^3 c_{3,2} s^{-1} + \beta_\tau^4 (c_{4,3} s^{-1} + c_{4,2} s^{-2}) + \cdots \right]. \end{aligned} \tag{4.2}$$

Plugging '$\beta_\tau = 0$' into the right-hand side gives '$\beta_\tau = s$'. Taking '$\beta_\tau = s$' and plugging it back into the right-hand side then gives '$\beta_\tau = s + c_{2,1}s^2 + (c_{3,2} + c_{4,2})s^3 + \cdots$'; iterating in this way yields a series expansion for β_τ:

$$\beta_\tau = \sum_{n=1}^{\infty} \alpha_{n,\tau} s_n = s + c_{2,1} s^2 + \left(2c_{2,1}^2 + c_{3,2} + c_{4,2}\right) s^3 + \cdots.$$

When $\tau = \infty$, the $\alpha_{n,\tau}$ are exactly the α_n that appear in (1.1). Note that the formulae generated for the $\alpha_{n,\tau}$,

$$\alpha_{1,\tau} = 1, \quad \alpha_{2,\tau} = c_{2,1}, \quad \alpha_{3,\tau} = 2c_{2,1}^2 + c_{3,2} + c_{4,2}, \quad \ldots$$

only depend on τ through the values of the $c_{a,b}$.

Lemma 4.3. *With* $S_n := \{(n_{a,b}) \in \mathbb{N}^I : \sum_I b n_{a,b} = n - 1\}$,

$$\alpha_{n,\tau} = \sum_{(n_{a,b})\in S_n} \frac{\left[\sum_I a n_{a,b}\right]!}{\left[\prod_I n_{a,b}!\right]\left[1 + \sum_I (a-1) n_{a,b}\right]!} \prod_I c_{a,b}^{n_{a,b}}.$$

The big Σ in the formula for $\alpha_{n,\tau}$ is a sum indexed by the elements of the finite set S_n; each element $(n_{a,b})$ of S_n is a sequence indexed by I.

It is a corollary of lemma 4.3 that $\alpha_{n,\tau} = \alpha_n$ if the memory $\tau \geqslant 2n - 2$. This follows from the definition of S_n. The formula for $\alpha_{n,\tau}$ only depends on $c_{a,b}$ with $b \leqslant n - 1$. If $(a,b) \in I$ and $b \leqslant n - 1$, then $a \leqslant 2n - 2$. Recall that $c_{a,b}$ is defined in terms of laces (and the corresponding compatible walks) of length a.

Proof of lemma 4.3. Let $\phi(\beta) = 1 + \sum_I c_{a,b} \beta^a s^{b-a}$. Setting $\beta = \sum_{n=1}^{\infty} \alpha_{n,\tau} s^n$, (4.2) becomes

$$\frac{\beta}{\phi(\beta)} = s.$$

The Lagrange–Bürmann series reversion formula [15, theorem 1.2.4] states that

$$\beta = \sum_{k=1}^{\infty} \frac{s^k}{k} [\beta^{k-1}] \phi(\beta)^k.$$

Applying the formula yields

$$\sum_{n=1}^{\infty} \alpha_{n,\tau} s^n = \sum_{k=1}^{\infty} \frac{s^k}{k} [\beta^{k-1}] \left(1 + \sum_I c_{a,b} \beta^a s^{b-a} \right)^k. \tag{4.4}$$

Let $T_k = \{(n_{a,b}) \in \mathbb{N}^I : \sum_I n_{a,b} \leqslant k\}$; by the multinomial theorem,

$$\left(1 + \sum_I c_{a,b}\beta^a s^{b-a}\right)^k = \sum_{(n_{a,b})\in T_k} \frac{k!}{\left[\prod_I n_{a,b}!\right]\left(k - \sum_I n_{a,b}\right)!} \prod_I \left[c_{a,b}\beta^a s^{b-a}\right]^{n_{a,b}}.$$

Extracting the coefficient of β^{k-1} from the right-hand side leaves only the terms corresponding to $(n_{a,b})$ in $U_k := \{(n_{a,b}) \in T_k : k-1 = \sum_I a n_{a,b}\}$. From (4.4) we obtain

$$\sum_{n=1}^{\infty} \alpha_{n,\tau} s^n = \sum_{k=1}^{\infty} \frac{s^k}{k} \sum_{(n_{a,b})\in U_k} \frac{k!}{\left[\prod_I n_{a,b}!\right]\left(k - \sum_I n_{a,b}\right)!} \prod_I [c_{a,b} s^{b-a}]^{n_{a,b}}$$

and so

$$\alpha_{n,\tau} = [s^n] \sum_{k=1}^{\infty} \sum_{(n_{a,b})\in U_k} \frac{s^k (k-1)!}{\left[\prod_I n_{a,b}!\right]\left(k - \sum_I n_{a,b}\right)!} \prod_I [c_{a,b} s^{b-a}]^{n_{a,b}}.$$

Extracting the coefficient of s^n on the right-hand side leaves only the terms with $n = k + \sum_I (b-a) n_{a,b}$. By the definition of U_k, these are the terms with $(n_{a,b}) \in S_n$. $\square$

5. Factorial bounds on $(\alpha_{n,\tau})$

We can use lemma 4.3 to bound the coefficients $(\alpha_{n,\tau})$ of the asymptotic expansions.

Lemma 5.1. *There is a constant C_2 such that $|\alpha_{n,\tau}| \leqslant C_2^n n!$.*

This is achieved by bounding $|c_{a,b}|$ in terms of b.

Lemma 5.2. *Let $c_b = \sum_{a=b+1}^{2b} |c_{a,b}|$. There is a constant C_3 such that $c_b \leqslant C_3^b b!$.*

Proof. The numbers $(c_{a,b})$ are defined in terms of laces with the length a:

$$|c_{a,b}| \leqslant \sum_{D=a-b}^{\lfloor a/2\rfloor} \sum_{N=1}^{\infty} f_\tau(a, N, D) \times |[s^{b-a}] s^{-1}(s^{-1}-2)\cdots(s^{-1}-2D+2)|.$$

The number of walks of length a in $\mathbb{Z}^D$ is $(2D)^a$, so that

$$\sum_{N=1}^{\infty} f_\tau(a, N, D) \leqslant \frac{(2D)^a}{2^D D!}.$$

The absolute value of $[s^{b-a}] s^{-1}(s^{-1}-2)\cdots(s^{-1}-2D+2)$ is at most

$$[s^{b-a}](s^{-1}+2D)^D = (2D)^{D+b-a}\binom{D}{a-b}, \qquad D = a-b, \ldots, \lfloor a/2\rfloor.$$

Therefore (as $D \leqslant \lfloor a/2\rfloor \leqslant b$),

$$|c_{a,b}| \leqslant \sum_{D=a-b}^{\lfloor a/2\rfloor} \frac{(2D)^a}{2^D D!}(2D)^{D+b-a}\binom{D}{a-b} = \sum_{D=a-b}^{\lfloor a/2\rfloor} \frac{2^b D^{D+b}}{(a-b)!(D-a+b)!}$$
$$\leqslant (1 + \lfloor a/2\rfloor - (a-b)) \frac{2^b b^{2b}}{(a-b)!(2b-a)!}.$$

By Stirling's approximation:

$$\frac{b^{2b}}{(a-b)!(2b-a)!} \leqslant \frac{(b!e^b)^2}{(a-b)!(2b-a)!} \leqslant \binom{b}{a-b} e^{2b} b! \leqslant 2^b e^{2b} b!.$$

Hence for some constant C_3,

$$c_b = \sum_{a=b+1}^{2b} |c_{a,b}| \leqslant \sum_{a=b+1}^{2b} (1 + \lfloor a/2 \rfloor - (a-b)) \cdot 2^b \cdot 2^b e^{2b} b! \leqslant C_3^b b!. \qquad \square$$

Before we can prove lemma 5.1, we need a bound on how a power series with factorial coefficients behaves under exponentiation.

Lemma 5.3. *Let $\phi(\beta) \equiv \sum_{k=0}^{\infty} k!\beta^k$. Then*

$$[\beta^k]\phi(\beta)^n \leqslant k! \prod_{j=1}^{k} (1 + (n-1)/j^2) \leqslant 6^n k!.$$

Before proving lemma 5.3, we will state a corollary that will be needed in section 7.

Corollary 5.4. *With C being a positive constant, let $\psi(\beta) \equiv \sum_{k=1}^{\infty} C^k \beta^k k!$ For $k \geqslant n$,*

$$[\beta^k]\psi(\beta)^n \leqslant (6C)^k (k-n)!.$$

Proof of corollary 5.4. For all m, $(m+1)! \leqslant 2^m m!$ and so

$$[\beta^k]\psi(\beta)^n \leqslant [\beta^k](C\beta\phi(2C\beta))^n = C^k 2^{k-n} [\beta^{k-n}]\phi(\beta)^n \leqslant C^k 2^{k-n} 6^n (k-n)!. \qquad \square$$

Proof of lemma 5.3. Let $l_1, \ldots, l_n$ denote non-negative integers. The first inequality is equivalent to

$$\sum_{l_1+\ldots+l_n=k} \prod_{i=1}^{n} l_i! \leqslant k! \prod_{j=1}^{k} (1 + (n-1)/j^2). \tag{5.5}$$

We will show this by induction in k. For convenience, (5.5) can be written in terms of a multinomial random variable $X^k \equiv \left(X_1^k, \ldots, X_n^k\right) \sim \text{Multinomial}(k; 1/n, \ldots, 1/n)$:

$$n^k \mathbb{E}\left(\binom{k}{X^k}^{-2}\right) \leqslant \prod_{j=1}^{k} (1 + (n-1)/j^2), \qquad \binom{k}{X^k} = \frac{k!}{X_1^k! \ldots X_n^k!}.$$

For the inductive step, we construct X^{k+1} from X^k by adding 1 to one of $X_1^k, \ldots, X_n^k$ uniformly at random. Let $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, and so on. The inductive step is then

$$\begin{aligned} n\mathbb{E}\left[\binom{k+1}{X^{k+1}}^{-2}\right] &= \mathbb{E} \sum_{i=1}^{n} \binom{k+1}{X^k + e_i}^{-2} \\ &= \mathbb{E} \sum_{i=1}^{n} \left(\frac{k+1}{X_i^k + 1}\right)^{-2} \binom{k}{X^k}^{-2} \\ &\leqslant \frac{(k+1)^2 + (n-1)}{(k+1)^2} \mathbb{E}\left[\binom{k}{X^k}^{-2}\right]. \end{aligned}$$

The inequality is the result of replacing $\sum_{i=1}^{n} \left(\frac{k+1}{X_i^k+1}\right)^{-2}$ with its supremum over the range of X^k.

The second inequality in the statement of lemma 5.3 follows from a well-known result of Euler: $\sum_{j=1}^{\infty} 1/j^2 = \pi^2/6$, and so

$$\log \prod_{j=1}^{k} \left(1 + (n-1)/j^2\right) \leqslant \sum_{j=1}^{k} \frac{n-1}{j^2} < (n-1)\pi^2/6 < \log(6^n). \qquad \square$$

Proof of lemma 5.1. For $(a, b) \in I$, $a \leqslant 2b$. By lemma 4.3,

$$|\alpha_{n,\tau}| \leqslant \sum_{(n_{a,b})\in S_n} \frac{\left(\sum a n_{a,b}\right)!}{\prod n_{a,b}!\left(1+\sum(a-1)n_{a,b}\right)!} \prod |c_{a,b}|^{n_{a,b}}$$

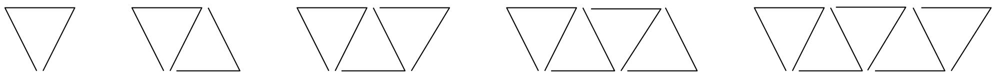
$$\leqslant \sum_{(n_{a,b})\in S_n} \frac{(2n-2)!}{\prod n_{a,b}!\left(2n-1-\sum n_{a,b}\right)!} \prod |c_{a,b}|^{n_{a,b}}$$

$$\leqslant \frac{1}{2n-1}[\beta^{n-1}]\left(1+\sum_{b=1}^{\infty} c_b \beta^b\right)^{2n-1}.$$

By lemma 5.2 and lemma 5.3, $|\alpha_n| \leqslant 6^{2n} C_3^n n!$. □

6. Diagrammatic estimates

The walks compatible with type-N laces can be represented by diagrams containing N segments, with each segment building a new loop. For $N = 1, 2, 3, 4$ and 5 we have



Such pictures have inspired a number of simple yet effective bounds on the lace expansion. In particular, there is a number C_{HS} such that for sufficiently large d , for all τ, [7]

$$\hat{\Pi}^{(N)}_{\beta_\tau}(0; \tau) \leqslant (s C_{\text{HS}})^N. \tag{6.1}$$

The method of diagrammatic estimates can be used to bound $\hat{\pi}_a^{(N)}(0; \tau)$, the number of walks of length a compatible with type-N laces.

Lemma 6.2. *There is a constant C_4 such that*

$$\hat{\pi}_a^{(N)}(0; \tau) \leqslant [\beta^a]\left(\sum_{n=1}^{\tau/2} C_4^n s^{-n} n! \beta^{2n}(1+s/\beta)\right)^N.$$

Proof. Let $f(n, x) = n\, c_n^{(0)}(x)$, where $c_n^{(0)}(x)$ denotes the number of simple walks from 0 to x of length n. Let $g(n) = \sup_x f(n, x)$. The loops corresponding to memory-τ laces have length at most τ. Consider the function

$$G^\tau(\beta) = \sum_{n=1}^{\tau} \beta^n g(n).$$

We will show, by the method of diagrammatic estimates, that

$$\hat{\pi}_a^{(N)}(0; \tau) \leqslant [\beta^a]\,(G^\tau(\beta))^N. \tag{6.3}$$

Let $(s_1, t_1), \ldots, (s_N, t_N)$ represent a typical lace of type N and length a. Let $t_0 = 0$. Note that

(i) $t_1 - t_0, t_2 - t_1, \ldots, t_N - t_{N-1} \in \{1, 2, \ldots, \tau\}$, and
(ii) $0 \leqslant s_{i+1} - t_{i-1} < t_i - t_{i-1}$ for $i = 1, \ldots, N-1$.

Let $\omega = (\omega(0), \omega(1), \ldots, \omega(a))$ represent a typical simple walk from 0 of length a. Then

$$\hat{\pi}_a^{(N)}(0; \tau) = \sum_{(t_i)} \sum_{(s_i)} \big|\{\omega : \omega \text{ is compatible with } (s_1, t_1), \ldots, (s_N, t_N)\}\big|.$$

The first sum is over the values of $t_1, \ldots, t_N$ compatible with (i). The second sum is over the values of $s_1, \ldots, s_N$ compatible with (ii). Take $t_1, \ldots, t_N$ to be fixed. Suppose that for some

$k = 1, \ldots, N$, we have fixed $s_1, \ldots, s_k$ and $\omega(1), \ldots, \omega(t_{k-1})$. How many ways are there to pick s_{k+1} and $\omega(t_{k-1}+1), \ldots, \omega(t_k)$?

The choice of $\omega(t_{k-1}+1), \omega(t_{k-1}+2), \ldots, \omega(t_k)$ is constrained by the requirement that $\omega(t_k) = \omega(s_k)$. The number of choices for the value of s_{k+1} is $t_k - t_{k-1}$. The total number of choices is at most $g(t_k - t_{k-1})$. Therefore,

$$\hat{\pi}_a^{(N)}(0;\tau) \leqslant \sum_{(t_i)} g(t_1 - t_0) g(t_2 - t_1) \cdots g(t_N - t_{N-1}),$$

and (6.3) follows.

Let $C_4 = 1000$. The lemma follows from (6.3) when we show that for $n = 1, \ldots, \tau$,

$$g(n) = \sup_x f(n,x) \leqslant [\beta^n] \sum_{n=1}^{\tau/2} C_4^n s^{-n} n! \beta^{2n}(1 + s/\beta) = C_4^{\lceil n/2 \rceil} s^{-\lfloor n/2 \rfloor} \lceil n/2 \rceil !.$$

First consider $n = 2m \leqslant 2d$. A walk from 0 to 0 in $\mathbb{Z}^d$ of length $2m$ has dimensionality at most m. The number of ways to pick an m-dimensional subspace of $\mathbb{Z}^d$ is at most $d^m/m!$. Using Stirling's formula

$$f(2m,0) \leqslant (2m)\frac{d^m}{m!}(2m)^{2m} \leqslant C_4^m s^{-m} m!.$$

For $x \neq 0$, let $i \geqslant 1$ denote the dimensionality of x. The number j of extra dimensions a walk from 0 to x of length $2m \leqslant 2d$ can explore is at most $m-1$, and the total dimensionality of the walk is at most $i + j \leqslant 2m$:

$$f(2m,x) \leqslant (2m)\frac{d^{m-1}}{(m-1)!}(2 \cdot 2m)^{2m} \leqslant C_4^m s^{1-m} m!.$$

Now consider $n = 2m+1$ for $1 \leqslant m < d$. Summing over the neighbours of x, $c_{2m+1}^{(0)}(x) = \sum_{y \sim x} c_{2m}^{(0)}(y)$. At most 1 of the $2d$ neighbours of x can be 0, hence

$$\begin{aligned} f(2m+1,x) &\leqslant \binom{2m+1}{2m}\left[C_4^m s^{-m} m! + (2d-1)\cdot C_4^m s^{1-m} m!\right] \\ &\leqslant C_4^{m+1} s^{-m}(m+1)!. \end{aligned}$$

Lastly, if $n \geqslant 2d$, $c_n^{(0)}(x) \leqslant (2d)^n$. Again by Stirling's formula

$$f(n,x) \leqslant n(2d)^n \leqslant C_4^{\lceil n/2 \rceil} s^{-\lfloor n/2 \rfloor} \lceil n/2 \rceil !. \qquad \square$$

7. Proof of theorem 1.3

It is well known that $d \leqslant \mu \leqslant 2d$, and hence $s \leqslant \beta_c \leqslant 2s$. Let $C_5 \in [0, C_2^{-1}]$ and suppose that $C_1 \geqslant 10C_2/C_5$. Then for $s \geqslant C_5/M$, inequality (1.3) holds simply by lemma 5.1. We will show by induction that for some positive constants C_5 and C_6, for $k = 0, 1, 2, \ldots$ and $s \leqslant C_5/k$,

$$\beta_c = \sum_{n=1}^{k} \alpha_n s^n + E_{k+1} s^{k+1} \qquad \text{with} \qquad |E_{k+1}| \leqslant C_6^{k+1}(k+1)!. \tag{7.1}$$

Theorem 1.3 follows from (7.1) by taking $C_1 = \max\{C_6, 10C_2/C_5\}$.

To begin the induction process, note that $E_1 = \beta_c/s \in [0,2]$, so (7.1) holds for $k = 0$ if $C_6 \geqslant 2$. As the proof progresses, we will impose a number of conditions on the pair of constants (C_5, C_6). The reader will see that all these conditions can be satisfied by first taking $C_6 \gg 1$ and then taking $C_5 \ll C_6^{-3}$.

Assume inductively that (7.1) holds for $k = 1, \ldots, M-1$. Fix $s \leqslant C_5/M$. We need to show that

$$|E_{M+1}| \leqslant C_6^{M+1}(M+1)!.$$

We will define $(A_i)_{i=1}^4$ such that

$$E_{M+1}s^M = \sum_{i=1}^4 A_i, \qquad |A_i| \leqslant \frac{1}{4}s^M C_6^{M+1}(M+1)!. \tag{7.2}$$

We will now use the lace expansion. We have assumed that $s = 1/(2d) \leqslant C_5/M$. If C_5 is sufficiently small, (4.1) holds for all τ.

A type-N memory-τ lace consists of N overlapping intervals. Each interval has length at most τ, so the total length is at most $N\tau$. It is therefore easier to use the finite memory version of the lace expansion. Recall that if $\tau \geqslant 2M-2$, the first M coefficients of the series expansions for β_c and β_τ agree: for $n = 1, \ldots, M$, $\alpha_n = \alpha_{n,\tau}$. Let $\tau = 2M$. By [6, theorem 1] there is a constant C_K such that

$$\forall s \leqslant 1/(52M),\ 0 \leqslant \beta_c - \beta_\tau \leqslant s^{M+2}C_K^M M!. \tag{7.3}$$

If $C_6 \geqslant C_K$, then $A_1 := \beta_c - \beta_\tau$ satisfies (7.2). Let $E_M^\tau = E_M - A_1$. Then

$$\beta_\tau = \sum_{n=1}^{M-1} \alpha_n s^n + E_M^\tau s^M. \tag{7.4}$$

By (4.1), we must now choose $A_2,\ A_3,\ A_4$ such that

$$A_2 + A_3 + A_4 + \sum_{n=1}^{M} \alpha_n s^{n-1} = 1 - \hat{\Pi}_{\beta_\tau}(0;\tau). \tag{7.5}$$

With reference to the diagrammatic estimate (6.1), let

$$A_2 = -\sum_{N=M+1}^{\infty} (-1)^N \hat{\Pi}_{\beta_\tau}^{(N)}(0;\tau).$$

If $C_5 \leqslant 1/(2C_{HS})$ and $C_6 \geqslant 2C_{HS}$, then A_2 satisfies (7.2).

Let A_3 match the terms generated on the right-hand side of (7.5) by the laces of length $a \geqslant 2M$ and type $N \leqslant M$,

$$A_3 = -\sum_{N=1}^{M}(-1)^N \sum_{a=2M}^{2MN} \hat{\pi}_a^{(N)}(0;\tau)\beta_\tau^a.$$

By lemma 6.2,

$$\begin{aligned} |A_3| &\leqslant \sum_{N=1}^{M}\sum_{a=2M}^{2MN} \beta_\tau^a[\beta^a]\left(\sum_{n=1}^{M} C_4^n s^{-n} n!\beta^{2n}(1+s/\beta)\right)^N \\ &\leqslant \sum_{N=1}^{M}(1+s/\beta_\tau)^N \sum_{a=2M}^{2MN} \beta_\tau^a[\beta^a]\left(\sum_{n=1}^{M} C_4^n s^{-n} n!\beta^{2n}\right)^N. \end{aligned}$$

Setting $x = \beta^2$, the right-hand side is equal to

$$\sum_{N=1}^{M}(1+s/\beta_\tau)^N \sum_{a=M}^{MN}\left(C_4 M\beta_\tau^2 s^{-1}\right)^a [x^a]\left(\sum_{n=1}^{M} n!x^n/M^n\right)^N.$$

If C_5 is small and C_6 is large, then $(1+s/\beta_\tau)C_4M\beta_\tau^2s^{-1} \leqslant C_6e^{-1}Ms \leqslant 1$, and

$$|A_3| \leqslant \sum_{N=1}^{M}\sum_{a=M}^{MN}(C_6e^{-1}Ms)^a[x^a]\left(\sum_{n=1}^{M}x^n n!/M^n\right)^N$$
$$\leqslant (C_6e^{-1}Ms)^M\sum_{N=1}^{M}\left(\sum_{n=1}^{M}n!/M^n\right)^N \leqslant \frac{1}{4}s^MC_6^{M+1}(M+1)!.$$

By the process of elimination, A_4 is now defined by

$$A_4+\sum_{n=1}^{M}\alpha_ns^{n-1} = 1-\sum_{N=1}^{M}(-1)^N\sum_{a=2}^{2M-1}\hat{\pi}_a^{(N)}(0;\tau)\beta_\tau^a. \tag{7.6}$$

Substitute (7.4) into the right-hand side of (7.6); by lemma 4.3, we can cancel the powers of s below s^M:

$$A_4 = -\sum_{n=M}^{\infty}s^n[s^n]\sum_{N=1}^{M}(-1)^N\sum_{a=2}^{2M-1}\hat{\pi}_a^{(N)}(0;\tau)\left(\sum_{n=1}^{M-1}\alpha_ns^n+E_M^\tau s^M\right)^a.$$

Recall that $c_b := \sum_{a=b+1}^{2b}|c_{a,b}|$. As $\alpha_1=1$ and $a\leqslant 2b$,

$$|A_4| \leqslant \sum_{n=M}^{\infty}s^n[s^n]\sum_{a=2}^{2M-1}\sum_{b=\lceil a/2\rceil}^{a-1}|c_{a,b}|s^{b-a}\left(\sum_{k=1}^{M-1}|\alpha_k|s^k+|E_M^\tau|s^M\right)^a$$
$$\leqslant \sum_{n=M}^{\infty}s^n[s^n]\sum_{b=1}^{2M-2}c_bs^{-b}\left(\sum_{k=1}^{M-1}|\alpha_k|s^k+|E_M^\tau|s^M\right)^{2b}. \tag{7.7}$$

Expand the $(\cdot)^{2b}$ terms on line (7.7) and then carry out the sum from $n=M$ to ∞: we will split the resulting terms into three groups.

(i) The terms $s^nc_b|\alpha_{i_1}||\alpha_{i_2}|\cdots|\alpha_{i_{2b}}|$ with $M\leqslant n\leqslant 2M-2$.
(ii) The terms $s^nc_b\ldots$ with $M\leqslant n\leqslant 2M-2$ that are not in group (i) because they contain an $|E_M^\tau|$ term.
(iii) The terms $s^nc_b\ldots$ with $n\geqslant 2M-1$.

Thinking of (4.2) as a recursive formula, lemma 5.1 is equivalent to the bound

$$|\alpha_{n+1}| \leqslant \left|[s^n]\sum_{b=1}^{\infty}c_bs^{-b}\left(\sum_{k=1}^{\infty}|\alpha_k|s^k\right)^{2b}\right| \leqslant C_2^{n+1}(n+1)!.$$

The contribution from the first group is therefore less than

$$\sum_{n=M}^{2M-2}s^nC_2^{n+1}(n+1)!. \tag{7.8}$$

The contribution of the second group is

$$\sum_{n=M}^{2M-2}s^n[s^n]\sum_{b=1}^{n+1-M}c_bs^{-b}\times(2b)\left(\sum_{k=1}^{M-1}|\alpha_k|s^k\right)^{2b-1}\left|E_M^\tau\right|s^M. \tag{7.9}$$

For $k=1,\ldots,M$, define $\hat{\alpha}_k$ by

$$\hat{\alpha}_ks^k = |\alpha_k|s^k+\ldots+|\alpha_{M-1}|s^{M-1}+\left|E_M^\tau\right|s^M.$$

We claim that the contribution of the third group is at most

$$s^{2M-1}[x^{2M-1}]\sum_{b=1}^{2M-2} c_b x^{-b}\left(\sum_{k=1}^{M}\hat{\alpha}_k x^k\right)^{2b}. \tag{7.10}$$

By having the $\hat{\alpha}_k x^k$ in the $(\cdot)^{2b}$ term, we catch all the terms in the third group while extracting only the coefficient of x^{2M-1}. [For example if $M = 10$, the term $s^{28}c_2\alpha_6\alpha_7\alpha_8\alpha_9$ is accounted for by the term $c_2 s^{-2}(\hat{\alpha}_6 s^6)(\hat{\alpha}_7 s^7)(\hat{\alpha}_7 s^7)(\hat{\alpha}_1 s) =$

$$c_2 s^{-2}(|\alpha_6|s^6+\cdots)(|\alpha_7|s^7+\cdots)(\ldots+|\alpha_8|s^8+\cdots)(\ldots+|\alpha_9|s^9+\cdots)$$

generated by (7.10).]

The absolute value of A_4 is now bounded by the sum of three intimidating expressions, (7.8)–(7.10). However, α_n are controlled by lemma 5.1, and c_b are controlled by lemma 5.2. By the inductive assumption and (7.3), if $C_6 \geqslant C_{\mathrm{K}}$ then $|E^\tau_M| \leqslant 2C_6^M M!$. If $C_6 \geqslant C_2$, we have $\hat{\alpha}_k \leqslant (2C_6)^k k!$. Substituting in these bounds and applying corollary 5.4 gives

$$\begin{aligned}|A_4| \leqslant \sum_{n=M}^{2M-2} s^n C_2^{n+1}(n+1)!\\ &+\sum_{n=M}^{2M-2} s^n \sum_{b=1}^{n+1-M}(C_3^b b!)\,(2b)\,(6\cdot C_2)^{n+b-M}(n+1-b-M)!\,\big(2C_6^M M!\big)\\ &+s^{2M-1}\sum_{b=1}^{2M-2}(C_3^b b!)(6\cdot 2C_6)^{2M+b-1}(2M-b-1)!.\end{aligned} \tag{7.11}$$

Let $k = n - M$. On the first line of (7.11), the $(n+1)!$ is less than $(M+1)!(2M)^k$; this turns the summand into a geometric series. On the second line, the summand (of the sum over b) is maximized by $b = k+1$, and the sum contains $k+1$ terms. On the third line, the summand is maximized by $b = 2M-2$, and the sum contains $2M-2$ terms. It follows that

$$\begin{aligned}|A_4| \leqslant s^M C_2^{M+1}(M+1)!\sum_{k=0}^{M-2}(s\cdot C_2\cdot 2M)^k\\ &+2s^M C_6^M M!\sum_{k=0}^{M-2} s^k(k+1)\big(C_3^{k+1}(k+1)!\big)(2k+2)(6C_2)^{2k+1}\\ &+s^{2M-1}(2M-2)\big(C_3^{2M-2}(2M-2)!\big)(12C_6)^{4M-3}.\end{aligned}$$

To complete the proof of theorem 1.3, we simply need to check that with $s \leqslant C_5/M$ and $M \geqslant 1$, $|A_4| \leqslant \frac{1}{4}s^M C_6^{M+1}(M+1)!$.

If C_5 is small, the two sums are dominated by their first terms; in particular, we can assume that each sum amounts to no more than twice the value of the summand when $k = 0$. As $(2M-2)\cdot(2M-2)! \leqslant (M+1)!(2M)^{M-2}$,

$$\frac{|A_4|}{s^M C_6^{M+1}(M+1)!} \leqslant 2\left(\frac{C_2}{C_6}\right)^{M+1} + \frac{48C_2C_3}{C_6(M+1)} + \frac{6\big(12^4\cdot 2MsC_3^2C_6^3\big)^{M-1}}{C_6 M}.$$

The right-hand side is smaller than $1/4$ if $C_6 \gg C_2C_3$ and $C_5 \ll C_3^{-2}C_6^{-3}$.

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References

[1] Madras N and Slade G 1993 The self-avoiding walk *Probability and Its Applications* (Boston, MA: Birkhäuser)
[2] Fisher M E and Gaunt D S 1964 Ising model and self-avoiding walks on hypercubical lattices and 'high-density' expansions *Phys. Rev.* **133** A224–39
[3] Gaunt D S and Ruskin H 1978 Bond percolation processes in d dimensions *J. Phys. A: Math. Gen.* **11** 1369–80
[4] Gaunt D S and Peard P J 2000 $1/d$-expansions for the free energy of weakly embedded site animal models of branched polymers *J. Phys. A: Math. Gen.* **33** 7515–39
[5] Gerber P R and Fisher M E 1974 Critical temperatures of classical n-vector models on hypercubic lattices *J. Math. Phys.* **10** 4697–703
[6] Kesten H 1964 On the number of self-avoiding walks. II *J. Math. Phys.* **5** 1128–37
[7] Hara T and Slade G 1995 The self-avoiding-walk and percolation critical points in high dimensions *Comb. Probab. Comput.* **4** 197–215
[8] Slade G 2006 *The Lace Expansion and Its Applications* (*Lecture Notes in Mathematics* vol 1879) ed Jean Picard (Berlin: Springer) (Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004) (DOI:10.1007/b128444)
[9] Clisby N, Liang R and Slade G 2007 Self-avoiding walk enumeration via the lace expansion *J. Phys. A: Math. Theor.* **40** 10973–1017
[10] Sokal A D 1980 An improvement of Watson's theorem on Borel summability *J. Math. Phys.* **21** 261–3
[11] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[12] de Gennes P G 1972 Exponents for the excluded volume problem as derived by the Wilson method *Phys. Lett.* A **38** 339
[13] Stanley H E 1968 Spherical model as the limit of infinite spin dimensionality *Phys. Rev.* **176** 718–22
[14] Brydges D and Spencer T 1985 Self-avoiding walk in 5 or more dimensions *Commun. Math. Phys.* **97** 125–48
[15] Goulden I P and Jackson D M 2004 *Combinatorial Enumeration* (Mineola, NY: Dover)